

An optimized rational fraction polynomial approach for modal parameters estimation from FRF measurements[†]

Omar Omar^{1,*}, Nejah Tounsi², Eu-Gen Ng¹ and M. A. Elbestawi¹

¹Department of Mechanical Engineering, McMaster University, Hamilton, ON, L8S 4L8, Canada

²AMTC-IAR-NRC, Montreal, QC, Canada

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Abstract

This paper presents an Optimized Rational Fraction Polynomial (ORFP) approach for modal parameters estimation from the measurements of the Frequency Response Function (FRF). Although this approach is based on the Rational Fraction Polynomial (RFP) technique described in [1], it suggests the use of a constrained optimization scheme rather than the Forsythe method to overcome the shortcomings of the Forsythe method. The latter are the estimation of modal parameters that do not necessarily describe a stable system and the estimation of fictitious natural frequencies. The formulation of the constrained optimization problem is presented and discussed. The assessment of the performance of the ORFP approach showed that it is better than the RFP approach in terms of its ability to identify modal parameters that ensure a stable system and its flexibility in selecting and setting the natural frequencies of the system. Several illustrative examples are given to demonstrate the robustness of the ORFP approach.

Keywords: Constrained optimization; Modal parameters extraction; Rational fraction polynomial; Stable system

1. Introduction

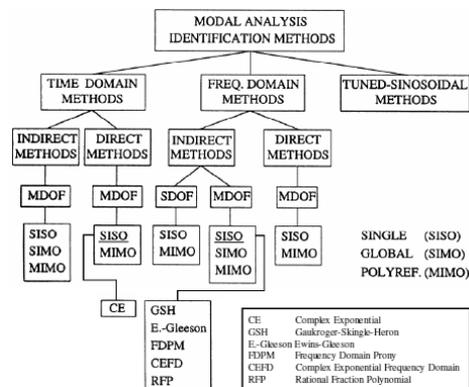
One of the fundamental areas of Experimental Modal Analysis is the Modal Parameters Extraction or Modal Parameters Identification. It refers to the identification of the dynamic properties of a given structure from the measured data. The dynamic properties are composed of the mass, the damping and the stiffness, which describe the dynamic equation of linear motion along u -direction as follows:

$$M_u \ddot{u} + C_u \dot{u} + K_u u = F_u \quad (1)$$

The parameters M_u , C_u and K_u can take a scalar form for a single-degree of freedom model or a matrix form for multi-degree of freedom models. The process of identifying the individual components of, M_u , C_u and K_u is simply a curve fitting process using different techniques of the model describing the Frequency Response Function (FRF) depending on the model FRF itself. Curve fitting techniques for modal parameters extraction can be divided into two main categories:

- Curve fitting in the frequency domain.
- Curve fitting in the time domain.

Each of those categories provides direct and indirect fitting techniques for single and multi-degree of freedom systems characterized by single input-single output (SISO), single input-multiple outputs (SIMO) or multiple inputs-multiple outputs (MIMO). The direct methods refer to the identification using the general equation of dynamic equilibrium. The indirect methods refer to identification using the modal parameters such as the natural frequencies and the damping ratios. A summary of these fitting techniques is provided in Fig. 1. Detailed information about these techniques could be found in [2].



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*Corresponding author. Tel.: +1 905 627 8912, Fax.: +1 905 527 7944

E-mail address: omaroe@gmail.com, omaroe@mcmaster.ca

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Fig. 1. Summary of curve fitting methods [3].

The selection of a fitting technique depends therefore on the dynamic system under consideration. The system under study could restrict the use of certain measurement instruments due to accessibility problem. Further more, the targeted area of the system to be studied could be small in size and cannot allow measurements at different points. Consequently the fitting techniques based on measuring the mode shape are not applicable in this case. This work, is focused on the identification of the dynamic parameters from a frequency response function, which is measured at a single point. This finds its application in machining where the small area of contact between the cutting tool and the work piece could be assimilated to a point contact, since this contact area is of the same order of magnitude with the contact area of the instruments used for the FRF measurements through the hammer impact testing. The resulting FRF in this study corresponds therefore to a SISO system and will include multi-degrees of freedom. The appropriate frequency-domain fitting techniques to this type of FRF are composed of: Gaukroger-Skingle-Heron method, Ewins-Gleeson method, Frequency Domain Prony method, the Complex Exponential Frequency Domain method and the Rational Fraction Polynomial (RFP). In the time domain, the appropriate fitting technique is the Complex Exponential method for curve fitting. The Gaukroger-Skingle-Heron method is based on a least-square fit of the receptance response of an N degrees of freedom system as given by Eq. (2).

$$\alpha(\omega) = \alpha_0 + \left(\sum_{r=1}^N \frac{A_r + i \omega B_r}{\omega_r^2 - \omega^2 + i 2 \xi_r \omega_r \omega} \right) e^{i\Phi} \quad (2)$$

where α_0 is a complex constant and Φ is an angle that represents the out-of-range modes. The model assumes viscous damping and is interactive allowing the user to take decisions such as the initial estimates for the modal parameters. An initial guess equal to zero for both α_0 and Φ was found to be satisfactory. A_r and B_r are obtained using a least-square analysis. Once A_r and B_r are known all the modal parameters are then known and the iterative process of minimizing a predefined error function can begin. Even though this method gives satisfactory results, the whole process is generally very slow [2].

The Ewins-Gleeson method is dedicated to identify the modal parameters for structures that are slightly damped (typically less than 10% [2] and in many cases less than 5% [4]) and the mathematical model is given by Eq. (3).

$$\alpha(\omega) = \sum_{r=1}^N \frac{C_r}{\omega_r^2 - \omega^2 + i \eta_r \omega_r^2} \quad (3)$$

where C_r is a real quantity due to the assumption of slightly damped system. This method works very well if the structure under investigation is in fact lightly damped. The main disadvantage is that this method is sensitive to selection of the FRF's point data to perform the curve fitting. To overcome

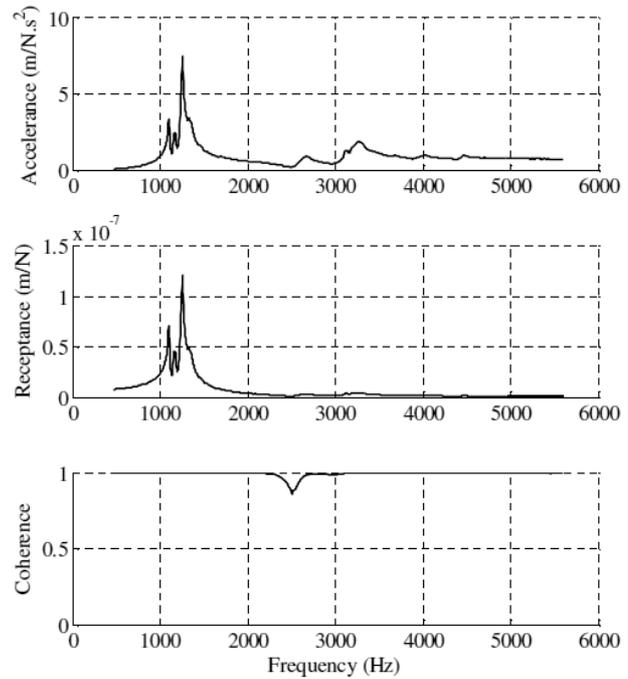


Fig. 2. Experimental FRF of the fixture used in the analysis.

this drawback, a different approach derived from the Rational Fraction Polynomial method was developed in [5]. In this new method, results are obtained with minimum intervention and experience of the user.

The Complex Exponential method is expressed in mathematical form in terms of displacement at point j due to a force applied at point k as given by Eq. (4).

$$\alpha_{jk}(\omega) = \sum_{r=1}^N \left(\frac{{}_r A_{jk}}{\omega_r \xi_r + i(\omega - \omega_r \sqrt{1 - \xi_r^2})} + \dots \frac{{}_r A_{jk}^*}{\omega_r \xi_r + i(\omega + \omega_r \sqrt{1 - \xi_r^2})} \right) \quad (4)$$

where ω_r is the natural frequency, ξ_r is the damping ratio and ${}_r A_{jk}$ is the residue corresponding to mode r and $(^*)$ denotes complex conjugate. The Complex Exponential method is designed to analyze a single response function. It is simple and doesn't require pre-estimation of the modal parameters. The only parameter that needs to be defined is the number of modes N for which the modal parameters will be extracted. The natural frequencies of the different modes ω_r , the damping factors ξ_r and the residues ${}_r A_{jk}$ can be determined through a series of matrix manipulations and operations. This process is further explained in details in [5, 6]. One of the major disadvantages of the Complex Exponential method is its sensitivity to noise [3, 7].

Tuned-sinusoidal methods are considered as a special class of modal identification methods which is based on the tuning of the real modes of vibration of a system by means of

excitation of the structure at each natural frequency by a set of exciters. This necessitate the pre-use of another identification method to know those natural modes and this is why some researchers do not consider it a genuine identification method [8].

The Rational Fraction Polynomial (RFP) method is one of the most popular and widely used multi-degree of freedom models in the frequency domain. It has been adopted and implemented by many commercial software packages of modal analysis. The FRF in terms of receptance can be given in partial fraction form as in Eq. (5).

$$\alpha(\omega) = \sum_{r=1}^N \frac{A_r + i\omega B_r}{\omega_r^2 - \omega^2 + i 2 \xi_r \omega_r \omega} \quad (5)$$

where A_r and B_r are constants and are called residues. The natural frequencies of different modes ω_r , the damping factors ξ_r and the residues A_r and B_r can be determined through a series of matrix manipulations and operations. This process is further explained in details in [5, 6]. The Rational Fraction Polynomial method was found to generate the most accurate damping ratio estimate [7] and provides very good estimates of the dynamic parameters if the FRF does not include noise, such the one obtained from analytical dynamic model. Since all the FRF resulting from measurements have inherently noise, the RFP will result in the estimation of modal parameters that do not necessarily describe a stable system and the estimation of fictitious natural frequencies. To overcome the shortcomings of the RFP method, this paper suggests the use of a constrained optimization scheme rather than the Forsythe method to estimate the modal parameters from measured FRF. The resulting approach, which is based on the RFP method, is named Optimized Rational Fraction Polynomial (ORFP) approach.

The remainder of this paper is organized as follows. Section 2 will focus on the investigation of the shortcomings of the RFP technique. Section 3 will introduce the ORFP approach and the formulation of the constrained optimization problem. Section 4 will present an investigation and an assessment of the performance of the ORFP approach. Section 5 will test the ORFP approach for robustness and versatility using some illustrative examples. Concluding remarks and prospective of future work are given in the last section.

2. Assessment of the rational fraction polynomial technique for modal parameters extraction

2.1 Overview of the RFP technique

The Frequency Response Function (FRF) is a complex function, which is defined over a specified frequency range. It can be obtained experimentally through an impact hammer test, which is often the easiest and quickest technique for measuring the FRF used in modal analysis [6]. The FRF resulting from an impact test is called accelerance when an

accelerometer is used to capture the output signal. Integrating the accelerance twice in the frequency domain generates the receptance. The choice of receptance in this work is related to our intention for the time domain simulation of the high speed machining, where the dynamic relative displacement between the tool and the workpiece will affect and will be affected by the instantaneous cutting forces. This simulation is beyond the scope of this paper. The system's modal parameters can be estimated by curve fitting a dynamic model such as the Rational Fraction Polynomial (RFP) model to the experimental receptance. A typical RFP model for a single-degree of freedom system is described in Eq. (6).

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{A}{i\omega - \lambda} + \frac{A^*}{i\omega - \bar{\lambda}} \quad (6)$$

where $H(\omega)$ is the receptance, $X(\omega)$ and $F(\omega)$ are the Fourier transforms of the displacement and the force respectively, ω is the frequency, $(^*)$ indicates the conjugate of a complex number, and λ is defined by Eq. (7).

$$\lambda = \sigma + i\omega_d \quad (7)$$

where ω_d is the damped natural frequency.

The FRF of a system with multi-degrees of freedom is obtained by summing the FRFs of all individual modes of relevance existing in the system as described by Eq. (8).

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \sum_{r=1}^N \left(\frac{A_r}{i\omega - \lambda_r} + \frac{A_r^*}{i\omega - \bar{\lambda}_r} \right) \quad (8)$$

where N is the total number of modes and $\omega_{d,r}$ is the damped natural frequency for mode r . λ_r is defined by Eq. (9).

$$\lambda_r = \sigma_r + i\omega_{d,r} \quad (9)$$

Note that Eqs. (8) and (5) are equivalent. In [1, 9, 10] a curve fitting technique was developed to extract the poles and the zeros (or residues) of an RFP model from the experimental FRF. This technique is based on the Forsythe method and the orthogonality of the polynomials in the RFP model. More details about the RFP method and the Forsythe method can be found in [1] and [11] respectively. The simplified Forsythe method presented by Richardson [1] was reported in [7] to be accurate, effective, and easy to use to extract the system's modal parameters. However, our investigation, which will be detailed hereafter, has revealed that some key issues in this method require improvement.

2.2 Investigation and assessment of the RFP technique

In [1], Richardson pointed out that in order to account for the effect of other modes or resonances outside the curve fitting band of interest the number of modes should be over specified. If the apparent number of modes is for example 4,

enhancing the curve-fitting results requires to specify a number of modes larger than 4, for example 9, to account for extra modes that do not show in the frequency spectrum of interest.

In our investigation it was found that over-specifying blindly the number of natural modes could result in the estimation of fictitious natural frequencies and the estimation of modal parameters that do not necessarily describe a stable system. In addition, Richardson pointed out that the orthogonality of the polynomials can break down when the frequency resolution is very low in the frequency band of interest, which is equivalent to under-sampling conditions.

To illustrate these shortcomings, let's consider the following case study where a fixture was subject to an impact hammer test. During this test, the structure starts to vibrate after receiving a blow from the hammer. The vibration will decay to rest after a certain period of time due to the structural damping existing in the system. The real part of the poles of an RFP model that describes such stable system is expected to be negative. The measured accelerance and coherence are plotted in Fig. 2 along with the evaluated receptance. These data are the results of the averaging of 5 measurements using Exponential windowing. The sampling frequency was equal to 16,384Hz. Data below 500Hz were ignored since the coherence of the signals is very low in this range, indicating that the data are not reliable.

When filtering the experimental FRF, it can lose modes but at the same time the curve fitting procedure might be enhanced resulting in a more stable system. The stability of the fitted FRF was investigated in light of filtering the experimental FRF using low-pass filters with different cut-off ratios to generate different frequency bands of interest. The cut-off ratio is identified as the ratio of the cut-off frequency to half the sampling rate and ranged from 0.0625 to 0.350. The corresponding cut-off frequencies ranged from 512Hz to 2867Hz. This range was selected to cover at least the major apparent dominant frequencies and the filtering took place before choosing the frequency range of interest.

In addition and for each frequency band, the number of natural modes used in the curve fitting process of the RFP model was varied from 8 to 40. The number of modes 8 was obtained by examining the measured FRF plotted in Fig. 2. The receptance FRF shows 3 dominant frequencies in the range of 1000-1500Hz and 3 other frequencies with very low amplitude close to 2800, 3300 and 4500Hz. In addition to these six modes, 2 other modes were considered to account for potential modes outside the range of frequency under consideration. The upper limit of investigating the number of modes, 40, was actually driven by the investigation itself since adding more modes was not of any benefit to the analysis as will be seen later.

The performance of the RFP technique is assessed in terms of the number of resulting stable modes and the goodness of fit. The latter is assessed by two criteria, which are the mean percentage error and the maximum percentage error between

the fitted and the experimental FRF. They are defined in Eqs.10 and 11 respectively.

$$C1 = \frac{1}{N} \sum_{n=1}^N \left| \frac{y_n - \hat{y}_n}{y_n} \right| \quad (10)$$

$$C2 = \max_{n \in \{1, \dots, N\}} \left[\frac{|y_n - \hat{y}_n|}{y_n} \right] \quad (11)$$

where n is the number of discrete frequencies in the frequency range, N is the total number of points in the record, y_n is the FRF estimate based on the RFP model and \hat{y}_n is the measured FRF after filtering.

The number of stable modes is plotted in Fig. 3 as function of the cut-off ratio and the total number of natural modes used for fitting. The number of stable modes is not significantly affected with the cut-off ratio. However, it increases as the number of natural modes increases. The ratio of the number of stable modes to the total number of modes increases abruptly from 15% up to approximately 50% as the total number of modes increases from 8 up to approximately 11, then it accues a drop with a minimum of 30% at 16 to increase again and reach a plateau at 55% near the total number of modes of 40, as shown in Fig. 4. It will be shown later on that with the increase of the number of total modes beyond a certain value, certain modes are estimated in pairs so that they cancel out each other and hence, the ratio asymptotes to a certain value. This gives an indication that the most appropriate number of natural modes should be in the neighborhood of 11, where the ratio of the number of stable modes to the total number of modes reach the first extremum. Having a number of modes larger than that will not enhance significantly the solution. In addition and since the number of stable modes did reach only 55% of the number of the total modes in the best case, the modal parameters extracted using the technique described by Richardson do not result in a stable system.

It was found that the mean percentage error decreases with the decrease of the cut-off ratio or with the increase of the number of suggested fitted modes as shown in Fig. 5. By decreasing the cut-off ratio, filtering the measured FRF results in higher loss of information and possibly loss of some apparent or closely spaced modes. Having fewer apparent natural modes to fit makes the process of curve fitting using

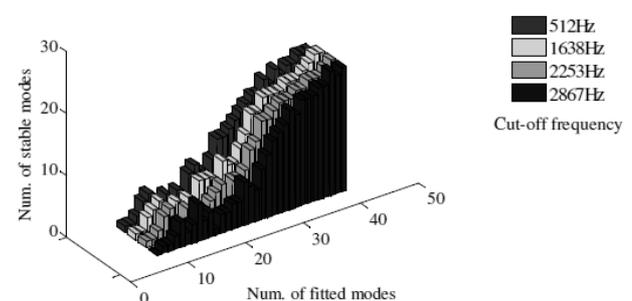


Fig. 3. Effect of the cut-off frequency and the total number of suggested fitted modes on the number of stable modes.

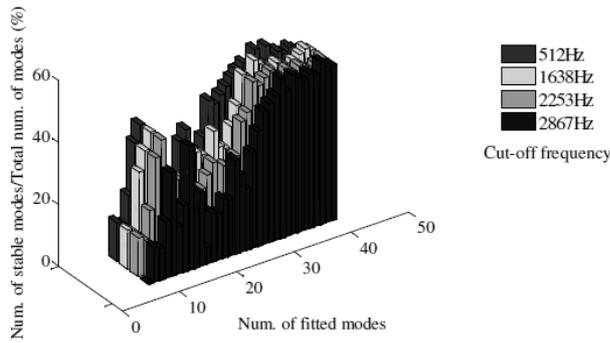


Fig. 4. Effect of the cut-off frequency and the total number of suggested fitted modes on the ratio of the number of stable modes to the total number of suggested fitted modes.

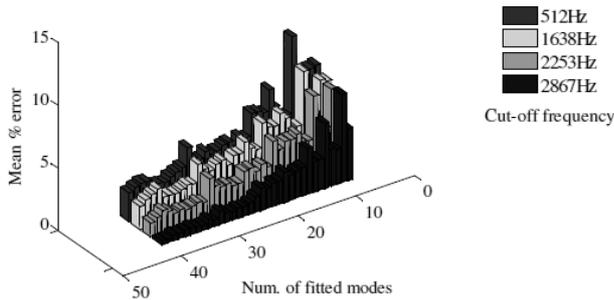


Fig. 5. Effect of the cut-off frequency and the total number of natural modes on the ratio of the mean percentage error of fitting the RFP model to the measured FRF

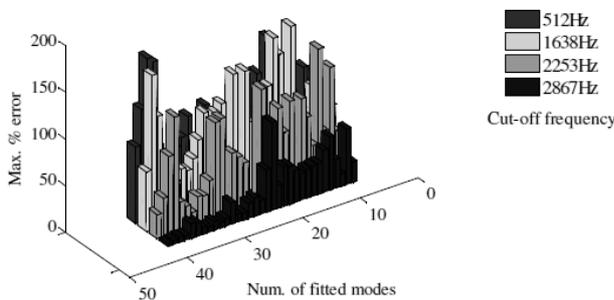


Fig. 6. Effect of the cut-off frequency and the total number of natural modes on the ratio of the maximum percentage error of fitting the RFP model to the measured FRF.

the RFP model easier and with smaller fitting errors. Moreover, the increase of the total number of modes generates pairs of poles with opposite signs that are estimated only for the sake of numerical balance. On the other hand, there is no clear trend for the maximum percentage error with either changing the cut-off ratio or the number of suggested fitted modes, as shown in Fig. 6.

The experimental FRF shown in Fig. 2 was filtered using a cut-off ratio of 0.0625, then fitted to 25 modes using the RFP model and the Forsythe method. The magnitude and phase of the filtered experimental and the fitted FRFs are plotted in Fig. 7. It is clear that filtering resulted in reducing the amplitude of

Table 1. Modal parameters for experimental data in Fig. 2 fitted to 25 modes.

Mode	Freq.(Hz)	$\xi\%$	Pole	
1	821.46	11.96	-621.62 +	5161.39i
2	1074.62	-4.54	306.55 +	6752.04i
3	1158.71	-8.33	608.76 +	7280.41i
4	1251.33	-4.06	319.46 +	7862.36i
5	1364.13	-4.39	376.94 +	8571.14i
6	1686.48	0.01	-0.01 +	10596.51i
7	2081.62	-3.90	510.94 +	13079.18i
8	2631.09	-3.17	524.91 +	16531.65i
9	2781.41	-1.83	320.29 +	17476.11i
10	3126.36	-2.72	533.84 +	19643.49i
11	3246.18	-2.07	423.28 +	20396.36i
12	3560.51	-0.41	90.81 +	22371.33i
13	3776.84	-0.89	212.71 +	23730.61i
14	3994.37	-1.96	494.08 +	25097.31i
15	4259.07	-0.49	132.43 +	26760.51i
16	4460.77	-1.27	358.44 +	28027.86i
17	4625.22	-0.69	202.66 +	29061.13i
18	4967.87	0.24	-76.25 +	31214.04i
19	4969.41	-1.51	471.53 +	31223.71i
20	5230.94	1.40	-460.97 +	32866.95i
21	5240.24	-1.14	374.04 +	32925.42i
22	5466.16	1.32	-454.61 +	34344.90i
23	5471.74	-1.14	393.33 +	34379.93i
24	5562.72	-0.17	59.58 +	34951.58i
25	5681.66	-0.38	136.49 +	35698.90i

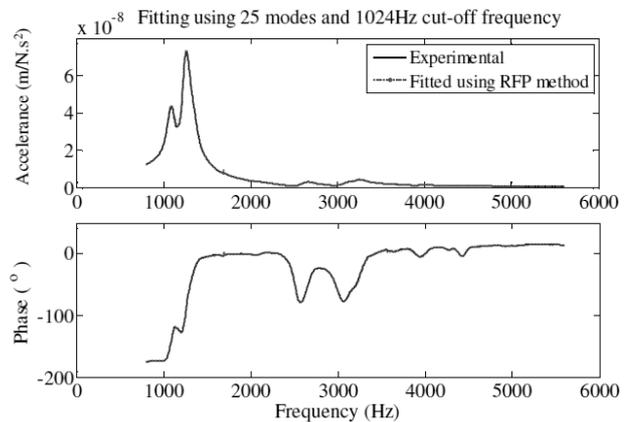


Fig. 7. FRF of the fixture fitted to 25 modes.

the FRF. For example the amplitude of the natural mode 1300Hz was reduced from 1.5 to 0.7 10^{-7} m/N, shifting some natural modes and losing one natural mode around 1200Hz.

A close match between the RFP model and the filtered experimental FRF can be observed for both the amplitude and the phase. However, an investigation of the estimated modal parameters, which are summarized in Tab.1, reveals that natural modes number 19 and 20 at 4969Hz and 5231Hz are hardly noticeable in the FRF plotted in Fig. 7, which suggests that they result only due to the least square fit of the Forsythe method. Similar conclusion can be also made about the pair of natural modes number 22 and 23. It was also noticed that the obtained natural frequencies do not necessarily line up with the experimental ones. For example, in the experimental FRF,

mode number 2 has a frequency of 1090Hz while in the fitted FRF it shows that it has a frequency of 1074Hz.

3. Formulation of the constrained optimization problem in the ORFP approach

A constrained optimization is formulated in this section to overcome the limitations of the RFP technique suggested by Richardson, which are the estimation of fictitious natural frequencies and the estimation of modal parameters that do not necessarily describe a stable system.

3.1 Formulation of the objective function

Several objective functions were formulated and were consequently assessed for better convergence. They include:

- Mean error, which is defined by Eq. (12):

$$Mean_e = \frac{1}{2N} \sum_{r=1}^N |Re(y(\omega_r)) - Re(\hat{y}(\omega_r))| + \dots \\ \frac{1}{2N} \sum_{r=1}^N |Im(y(\omega_r)) - Im(\hat{y}(\omega_r))| \quad (12)$$

where N is the total number of data points in the record.

- Maximum error, which is defined by Eq. (13):

$$Max_e = \max_{r \in \{1, \dots, N\}} \left(\frac{1}{2} |Re(y(\omega_r)) - Re(\hat{y}(\omega_r))| + \dots \right. \\ \left. \frac{1}{2} |Im(y(\omega_r)) - Im(\hat{y}(\omega_r))| \right) \quad (13)$$

- Mean percentage error, which is defined by Eq. (14):

$$Mean_{e,p} = \frac{50}{N} \sum_{r=1}^N \left| \frac{Re(y(\omega_r)) - Re(\hat{y}(\omega_r))}{Re(y(\omega_r))} \right| + \dots \\ \frac{50}{N} \sum_{r=1}^N \left| \frac{Im(y(\omega_r)) - Im(\hat{y}(\omega_r))}{Im(y(\omega_r))} \right| \quad (14)$$

- Max. percentage error, which is defined by Eq. (15):

$$Max_e = \max_{r \in \{1, \dots, N\}} \left(50 \left| \frac{Re(y(\omega_r)) - Re(\hat{y}(\omega_r))}{Re(y(\omega_r))} \right| + \dots \right. \\ \left. 50 \left| \frac{Im(y(\omega_r)) - Im(\hat{y}(\omega_r))}{Im(y(\omega_r))} \right| \right) \quad (15)$$

- Least square error, which is defined by Eq. (16):

$$LS_e = \left(\frac{0.5}{N} \sqrt{\sum_{r=1}^N [Re(y(\omega_r)) - Re(\hat{y}(\omega_r))]^2} \right) + \dots \\ \left(\frac{0.5}{N} \sqrt{\sum_{r=1}^N [Im(y(\omega_r)) - Im(\hat{y}(\omega_r))]^2} \right) \quad (16)$$

3.2 Assessment of the objective function

The performance of the different objective functions will be evaluated in terms of the occurrence of the convergence, the number of iterations required for convergence and the goodness of the fit. The latter is assessed by the following two criteria:

- **Criterion 1** is defined as the mean percentage error between the fitted and the experimental FRF and is defined by Eq. (10).

- **Criterion 2** is defined as the maximum percentage error between the fitted and the experimental FRF and is defined by Eq. (11).

For each objective function, the experimental FRF illustrated in Fig. 2 was fitted to the RFP model using the ORFP approach. The number of natural modes was set to 15 and the record set contained 480 data points. The maximum number of iterations was selected to be 2500 iterations and the cut-off frequency was 2048Hz.

At the end of the optimization process and to facilitate the comparison of the different objective functions, a normalization of these functions was required to harmonize their values in terms of magnitudes. The normalization is accomplished by dividing each objective function by its maximum value obtained over all the iterations required for the convergence of the optimization process. The maximum value of each objective function is generally obtained in the first iteration and consequently the normalized objective functions are expressed by Eq. (17).

$$\chi|_k = \frac{OF|_k}{OF_1} \quad (17)$$

where k is the iteration number and OF is the objective function under investigation.

The results on the occurrence of the convergence are reported in Tab.2. They indicate that, among all the objective functions, the maximum % error and the least square error did not lead the ORFP to converge. The objective functions that resulted in the convergence of the ORFP were plotted against the progressive number of iterations as illustrated in Fig. 8. The plot of each normalized objective function features a high decrease rate or high convergence rate at the starting of the optimization process, followed by a low decrease rate or low convergence rate, which might lead to a plateau corresponding to the convergence if the number of iterations is sufficiently high.

For the maximum error, the mean error and the mean percentage error, the corresponding slow convergence phases start at approximately 1100, 1400 and 2000 iterations and they are at the levels of 0.5, 0.1 and 0.15, respectively. Therefore, the ORFP based on the maximum error is the fastest to converge while the one based on the mean error is the slowest to converge.

The two criteria of the goodness of the fit will be used to

Table 2. Comparison of different objective function’s convergence.

Obj. fun.	Convergence	Num. of it ⁿ at convergence	χ after convergence	Goodness of fit *	
				C1	C2
Mean error	Yes	2000	0.15	1.66	31.24
Max error	Yes	1100	0.10	82.85	440.70
Mean % error	Yes	1400	0.05	5.05	137.23
Max % error	No	-	-	-	-
Least square error	No	-	-	-	-

*C1, C2 = Criterion 1 and Criterion 2 respectively.

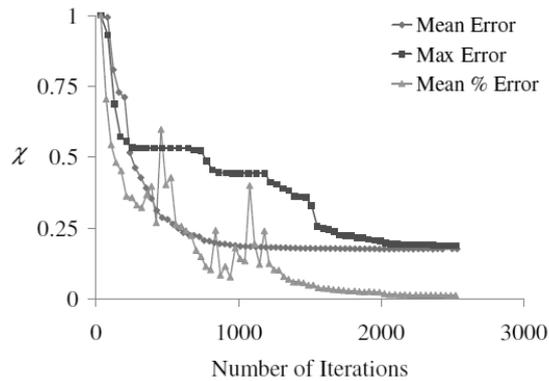


Fig. 8. Investigation of different objective functions.

select the most appropriate objective function. The corresponding values are listed in Tab.2. The objective function defined by the mean error resulted in the values of 1.66 and 31.23 for criteria 1 and 2, respectively. These values are at least 3 times lower than those achieved by the objective function defined by the mean percentage error and at least 14 times lower than those achieved by the objective function defined by the maximum error.

It becomes obvious that the objective function defined by the mean error provides the best fit at the expense of slower convergence, which means a higher number of iterations and consequently higher computing time.

3.3 Formulation of the constraints

The optimizer minimizes the cost function while maintaining the following constraints:

- The structural damping ratio ζ is commonly found to be less than 10% [3].

$$0\% < \zeta < 10\% \tag{18}$$

- The natural frequency of the system are maintained constant during the optimization process. They are selected based on the natural modes featured in the measured FRF.
- The real part of each pole is maintained negative to ensure that the estimated modal parameters describe a stable system.
- The optimization process stops after a certain number of iterations that has to be pre-determined before starting the optimization as this forms the stopping criterion of the whole process.

3.4 Convergence (Stopping) criteria

The optimization process stops if the specified number of iterations is reached or if the rate of variation of both the objective function and the solution are within specified tolerances. The selection of the number of iterations in this work is dictated by the goodness of the fit, which is defined by the two criteria mentioned above: the mean percentage error and the maximum percentage error. It was found that after a certain number of iterations, the goodness of the fit cannot be enhanced any further as the number of iterations increases while the objective function defined by the mean error could achieve smaller value. Selecting the number of iterations based on the tight convergence of the objective function will result in therefore increasing the computing time without enhancing the goodness of the fit. This can be illustrated through the following example. Using the same experimental FRF as before and the objective function defined by the mean error, the optimization process was conducted with different number of iterations, ranging from 200 up to 3,000. For each specific number of iterations, the two criteria for the goodness of the fit were evaluated and the results are shown in Fig. 9. This figure indicates that both criteria started at a relatively high value then converge to a plateau starting approximately at 2000 and 1400 iterations for criteria 1 and 2, respectively. These numbers of iterations are smaller than the number of iterations required by the objective function defined by the mean error to converge, which exceeds 2500 as per Fig. 8. It can be concluded that the convergence of this objective function guarantee the goodness of the fit. In addition, the specified tolerance on the rate of variation of the objective function and the criteria of the goodness of fit. Since there is a difference in the number of iterations required by the two criteria to converge, these two criteria need to be monitored at the same time to save unnecessary computational time.

3.5 Selection of the initial guess

Two alternatives were investigated to define the initial guess. The first one consists of using the solution obtained from Richardson's technique based on the RFP and the Forsythe method as the initial guess. The second alternative consists to have rough estimates based on the measurement of the FRF. It requires the evaluation/estimation of four values, which are:

1. The number of selected natural modes.
2. The natural frequencies corresponding to each desired mode, which can be obtained graphically from the experimental FRF.
3. The maximum absolute magnitude value of the experimental FRF.
4. The maximum structural damping ratio that will be used in the fit.

The poles which will form the initial guess can be defined as:

$$\text{Poles} = 2\pi f_{n,j} \left(\frac{-\xi_j}{100} + i \right) \text{ for } j \in \{1 \dots r\} \quad (19)$$

where $f_{n,r}$ is the natural frequency of mode r and the maximum structural damping ratio ξ_j can be chosen to be 4% for all modes. While the poles represent the position of the modes and how much they are spread (damping), the residues form the amplitude of the mode itself. Hence the residues which will form the initial guess can be chosen to be the maximum absolute magnitude value of the FRF to be fitted.

3.6 Coding the variables

In the optimization process, the residues and the poles of the complex rational polynomial equation need to be identified including both the real and imaginary parts. Consequently, there are 4 terms to be identified for each mode. In this work, the imaginary part of the pole, which represents the natural frequency, will be obtained directly from the experimental FRF. The other three terms, which are the real and imaginary parts of the residue and the real part of the pole, highly differ in their orders of magnitude. To enhance the robustness and the rate of convergence of the optimization process and to avoid ill-conditioned matrices, all the variables should be

harmonized in terms of magnitudes. In the objective function and in the constraints, each variable x will be expressed as the product of a modified variable x_{mod} by the order of magnitude. The vector of order of magnitudes will remain constant during the optimization process. However, the modified variables have similar order of magnitudes and will be treated as the unknown. Since the magnitude of each variable is not known before-hand, it is suggested to use the initial guess defined earlier as an indicative of the expected order of magnitudes for the different variables. The initial guess denoted x_o is coded as follows. The values in vector x_o are transformed to the scientific number format. The exponent of each number (the part after the e) indicates the order of magnitude of the variable. The different exponents are saved in a specific vector, which will be used in the objective function and the constraints to scale up or down the modified variables. The remaining part (mantissa) of the values in vector x_o have the same order of magnitude and constitutes the vector of modified variables denoted x_{mod} . This vector can be used as the initial guess for the optimizer. The final solution of the optimization process is obtained by decoding the modified variables through multiplying the modified variables by the vector of the order of magnitudes.

3.7 Optimizer

The Matlab™ function *fmincon* is used as the optimizer. The optimization process is subject to the minimization of the objective function while satisfying the constraints as described in Sec. 3.3. The input to the optimization algorithm are the initial guess of the poles and zeros, the number of iterations, the number of modes to be fitted, and the experimental FRF, which is the receptance. The receptance is obtained by integrating the acceleration twice in the frequency domain. The optimizer options, which include the number of iterations and the type of optimization engine to be used in the optimization process, were first set-up. The initial guess were then coded according to the procedure described previously then passed to the objective function and to the constraints. The optimization process stops after the prescribed number of iterations and the modified solution is decoded to obtain the final values of the dynamic parameters. It should be noted that during the analysis the initial guess was obtained at first through the RFP and Forsythe method then the improvement described in Sec.3.5 was implemented. In both ways, the poles' real number were always forced to be negative to insure the stability of the first iteration.

4. Assessment of the ORFP approach

The flexibility of the ORFP method compared to the use of the RFP and Forsythe method will be first demonstrated. Then, the ORFP method will be investigated for the effect of choosing different number of modes on the normalized objective function defined by the mean error and on the two criteria for the goodness of the fit, for the meaningfulness of

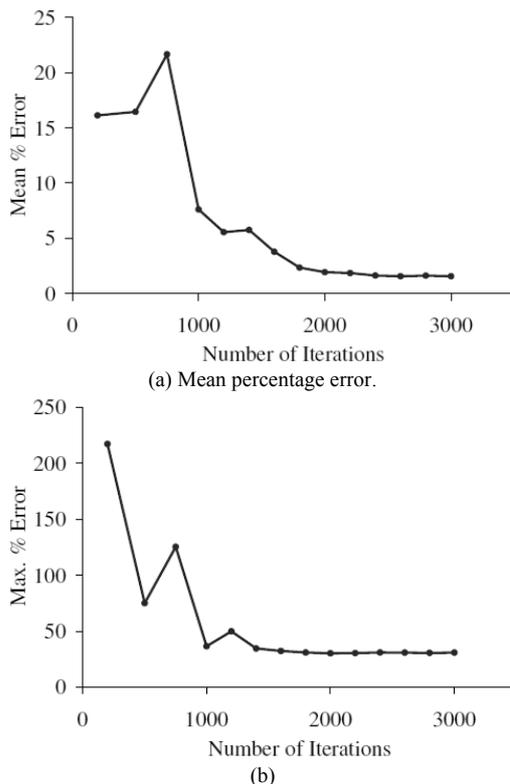


Fig. 9. Effect of varying the number of iterations on the mean and maximum percentage error between the fitted and experimental FRF respectively.

the suggested number of modes, and for the optimization process computational time.

4.1 Flexibility of the ORFP method

When using the Forsythe method with a fixed number of modes to be fitted, the resulting poles and zeros are directly obtained from a least square fit using matrices relations and orthogonally of polynomials. The estimated natural modes may exhibit a significant shift from the experimental ones. In addition to the estimation of some poles real parts with negative sign, which was mentioned earlier, it estimates relatively high values (close to 10%) for structural damping. Moreover, it predicts some fictitious modes that do not exist in the experimental FRF. To illustrate these conclusions, the experimental FRF was filtered using a cut-off frequency of 3072Hz and a frequency range of 800-2000Hz, then fitted to 7 modes using the RFP and Forsythe extraction method. The resultant modal parameters are listed in Tab.4 and the fitting curves are presented in Fig. 10 along with the experimental FRF. Tab.4 shows clearly the existence of 6 negative damping ratios. Therefore 6 out of 7 poles have negative real parts. The natural mode 1 at approximately at 900Hz has about 10% damping ratio, which is relatively high for structural damping. Modes 1 and 6 are not distinguishable on the experimental FRF. There is a frequency-shift between the estimated natural modes 2-5 and the corresponding experimental ones as shown in Fig. 10.

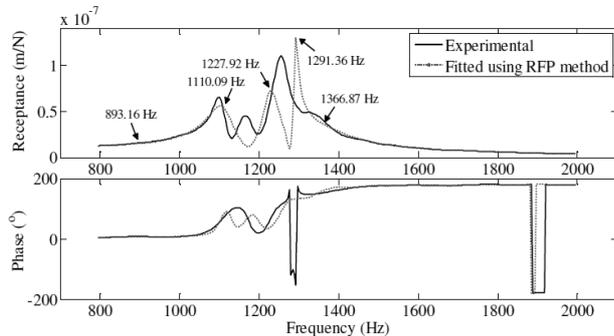


Fig. 10. Fixture FRF fitted to 7 modes using the Forsythe method.

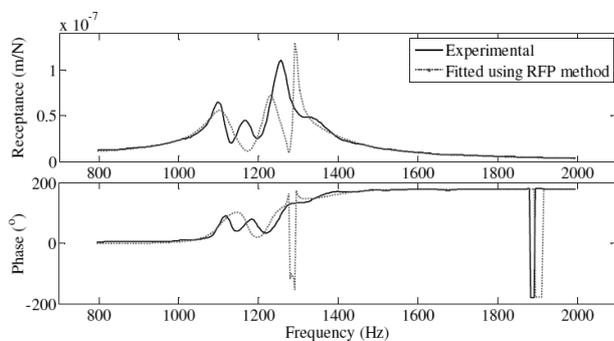


Fig. 11. FRF regenerated after eliminating modes 1 and 6.

To assess the effect of the fictitious modes 1 and 6 on the goodness of the fit, it is suggested to rebuild the FRF with only the remaining 5 modes (modes 1 and 6 are excluded) estimated by the RFP and Forsythe method. The results are presented in Fig. 11. The criteria for the goodness of the fit are also evaluated and listed in Tab. 3. Both criteria indicate that estimating the FRF by removing modes 1 and 6 is worse than the one where these two modes are considered.

To evaluate the effect of the frequency shift in the predicted natural modes compared to the experimental ones, it is suggested to estimate the FRF with the 5 modes (2-5 and 7) where the natural frequencies were set equal to those of the experimental FRF as per Tab. 3. The results are presented in Fig. 12. The criteria for the goodness of the fit are also evaluated and listed in Tab.3. Both criteria indicate that the estimating the FRF by removing modes 1 and 6 and matching the natural frequency of modes (2-5 and 7) to the experimental ones is worse than the original FRF estimate by the Forsythe method.

This indicates clearly that the selection of the natural modes, which can be obtained easily from the experimental FRF, cannot help improving the goodness of the fit when the RFP and Forsythe method are used.

However, the ORFP method offers the possibility and the flexibility to select the natural modes to be fitted and to fix their natural frequencies to the values obtained graphically from the experimental FRF. For the previous example, we can

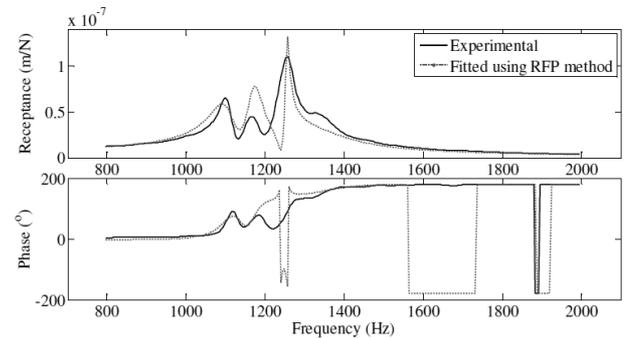


Fig. 12. FRF regenerated after eliminating modes 1 and 6 and modifying the remaining modes natural frequencies.

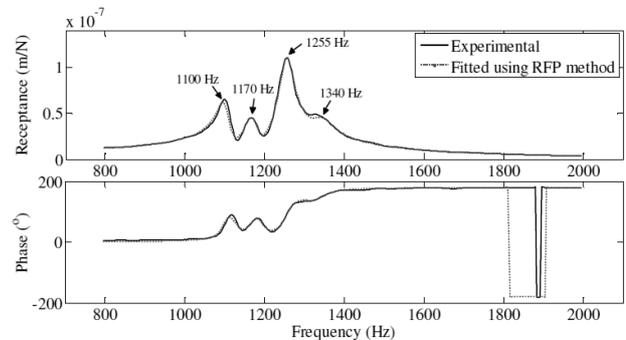


Fig. 13. FRF regenerated using the ORFP method after eliminating modes 1 and 6 and modifying the remaining modes.

fix the number of modes to 5, which is the number of apparent modes in the experimental FRF. The corresponding natural frequencies are listed in Tab.3.

Fixing the natural frequencies to those obtained graphically from the experimental FRF has an impact on the initial guess obtained through the first alternative described earlier, which is the RFP and Forsythe method. The imaginary part of the poles is modified to account for the desired natural frequencies. The sign of the poles' real part as well as the real and imaginary parts of the residues are kept unchanged.

The use of the ORFP with the resulting initial guess leads to the extraction of the modal parameters presented in Tab.4. The estimated and experimental FRFs are shown in Fig. 13. The criteria for the goodness of the fit are also evaluated and listed in Tab.3. Both criteria indicate a good estimation of the FRF using the ORFP method and the modified initial guess obtained by the RFP and Forsythe method. Moreover, the resulting damping ratios presented in Tab.4 are within the norms of structural damping (<10%). The real parts of the poles are all positive, which guarantees a stable response of the system. The estimation made by the ORFP method is

therefore much superior than the one provided by the RFP and Forsythe method.

The use of the initial guess obtained by the second alternative described earlier leads to the modal parameters listed in Table 4. The estimated and experimental FRFs are shown in Fig. 14. The criteria for the goodness of the fit are also evaluated and listed in Tab.3. Both criteria indicate a good estimation of the FRF using the ORFP method and the initial guess obtained by the second alternative. These criteria and the estimated modal parameters using the initial guesses obtained by both alternatives are almost identical. This indicates that there is no need to use the RFP and Forsythe method to estimate an initial guess. The latter can be easily obtained from the experimental FRF as described by the second alternative. This provides the ORFP with more flexibility in estimating modal parameters compared to the use of the RFP and Forsythe method.

4.2 Effect of choosing different number of modes on the normalized objective function defined by the mean error and on the two criteria for the goodness of the fit

Different numbers of modes have been used to estimate a fit for the experimental FRF data using the mean error as objective function. The experimental data were filtered using a cut-off frequency of 3072Hz. The number of iterations was fixed to 3000 iterations. The normalized objective functions are plotted in Fig. 15 against the number of iterations for the number of modes ranging from 8 to 15. All curves exhibit the same trend: a high rate of convergence at the beginning of the iterative optimization process then the rate of convergence decrease towards the end of the process where it can reach 0. Increasing the suggested number of modes to estimate the FRF increases the number of iterations required by the normalized objective function to converge. For instance, fitting the experimental FRF data to 15 modes takes 1800 iterations to converge, while 900 iterations were enough for the case where only 8 modes are used but this comes over the expense of the accuracy of the final fit. Increasing the number of modes will reduce the final values of the objective function and the two criteria for the goodness of the fit, as can be

Table 3. Summary of the investigation of the ORFP method's flexibility over the RFP-Forsythe method.

Scenarios	N. of modes	Nat. freq. (Hz)	Goodness of fit	
			C1*	C2*
Extraction of method using RFP-Forsythe method.	7	$f_{n1} = 893.16$ $f_{n2} = 1110.09$ $f_{n3} = 1227.92$ $f_{n4} = 1291.36$ $f_{n5} = 1366.87$ $f_{n6} = 1698.49$ $f_{n7} = 1902.84$	10	81.36
Estimation of the FRF based on specific 5 modes (2,3,4,5,7) obtained using RFP-Forsythe method.	5	$f_{n2} = 1110.09$ $f_{n3} = 1227.92$ $f_{n4} = 1291.36$ $f_{n5} = 1366.87$ $f_{n7} = 1902.84$	12	82.06
Estimation of the FRF based on specific 5 modes (2,3,4,5,7) obtained using RFP-Forsythe method and shifting modes (2, 5) to match the experimental ones.	5	$f_{n2} = 1110.09$ $f_{n3} = 1227.92$ $f_{n4} = 1291.36$ $f_{n5} = 1366.87$ $f_{n7} = 1902.84$	18	126.97
ORFP method an the modified initial guess obtained by the RFP and Forsythe method.	5	$f_{n2} = 1110.09$ $f_{n3} = 1227.92$ $f_{n4} = 1291.36$ $f_{n5} = 1366.87$ $f_{n7} = 1902.84$	2.88	21.22
ORFP method an the modified initial guess obtained by a rough estimate from the experimental FRF.	5	$f_{n2} = 1110.09$ $f_{n3} = 1227.92$ $f_{n4} = 1291.36$ $f_{n5} = 1366.87$ $f_{n7} = 1902.84$	3.12	22.20

*C1, C2 = Criterion 1 and Criterion 2 respectively.

Table 4. Comparison between RFP-Forsythe method and the ORFP method to show the flexibility of the later in predicting the damping ratio ξ .

Mode #	RFP-Forsythe method		ORFP method and the modified initial guess obtained by the RFP and Forsythe method		ORFP method and the modified initial guess obtained by a rough estimate from the experimental FRF	
	f_n (Hz)	ξ	f_n (Hz)	ξ	f_n (Hz)	ξ
1	893.16	-10.61	-	-	-	-
2	1110.09	- 3.01	1100	1.51	1100	1.51
3	1227.92	-1.62	1170	1.55	1170	1.55
4	1291.36	0.38	1250	1.6	1250	1.6
5	1366.87	-7.87	1340	2.44	1340	2.43
6	1698.49	-2.02	-	-	-	-
7	1902.84	-0.62	1903	3.84	1903	3.23

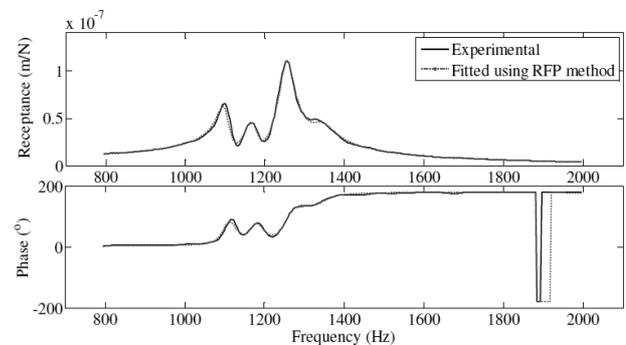


Fig. 14. FRF regenerated using the ORFP method after eliminating modes 1 and 6 and modifying the remaining modes to match the experimental ones and guessing the initial values of the damping.

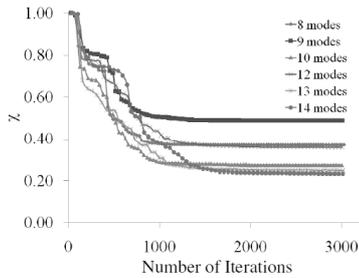


Fig. 15. Investigation of the number of modes on χ .

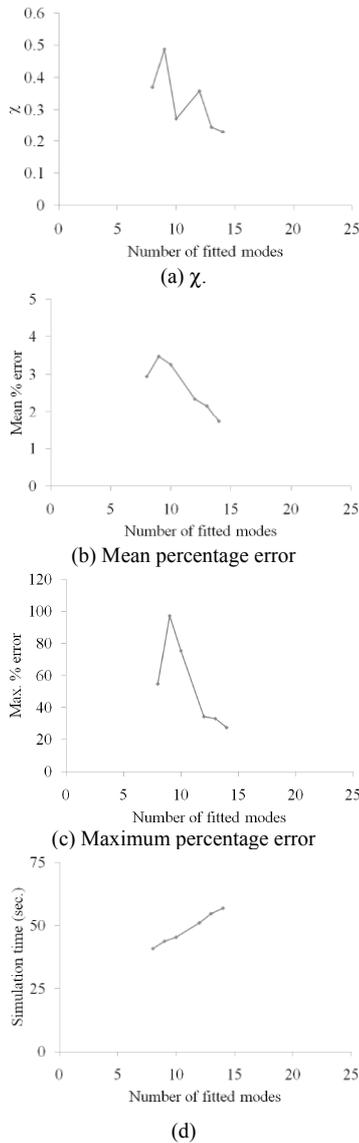


Fig. 16. Effect of varying the number of iterations on different performance parameters.

illustrated in Figs. 16(a), 16(b), 16(c) respectively. Increasing the number of modes from 8 to 14 has dropped the value of the normalized objective function from 0.37 to 0.23, the mean percentage error from 2.9% to 1.7% and the maximum percentage error from 54% to 27%.

Table 5. Summary of different FRFs case examples to show the robustness and versatility of the ORFP method.

Example	Opt. # of iter.	Opt. # of modes	Computing time	χ	Goodness of fit	
					C1	C2
1	1800	6	18.06	0.0411	3.07	14.23
2	1200	6	18.69	0.0934	4.94	37.44

*C1, C2 = Criterion 1 and Criterion 2 respectively.

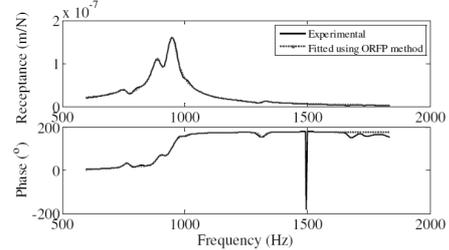


Fig. 17. Application of the ORFP method to the first example of measured FRF.

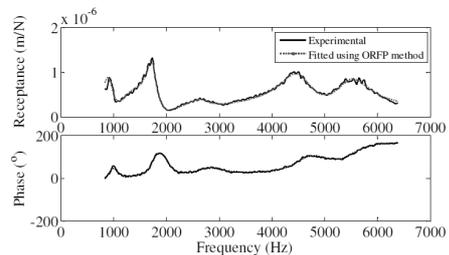


Fig. 18. Application of the ORFP method to the second example of measured FRF.

4.3 Optimization process computational time

As pointed out earlier, there is a trade off to be made between the number of modes and the goodness of the fit. This will have also an impact on the number of iterations and consequently the computing time, as illustrated in Fig. 16 (d). Increasing the number of modes from 8 to 14 results in an increase of the simulation time from 41 to 56 seconds.

5. Illustrative examples

To test the robustness and the versatility of the ORFP method, it was applied to additional two FRFs that describe two different mechanical structures, for instance a fixture in Example 1 and a cutting tool in Example 2.

The FRF of the first example is plotted in Fig. 17. It features two dominant and close modes in the range of 800-900Hz. The frequency spectrum covers from 500 to 2000Hz. However, the frequency spectrum of the FRF of the second example is 3 times wider and covers from 500 up to 6000Hz, as shown in Fig. 18. In addition, it has the distinct feature of having only three dominant and distinct modes in the neighbourhood of 2000, 4500 and 5500Hz, respectively and two other distinct modes in the neighbourhood of 800 and 2400Hz. It should be noted that the FRF of the fixture that was used previously has a combination of the characteristics of the FRF of the first example and the FRF of the second example

since it features both close modes and distinct modes. The results obtained by the ORFP are summarized in Tab.5 and the predicted and experimental FRF are plotted in Figs.17-18 for Ex. 1 and Ex. 2 respectively. The good agreement between the experimental and simulated FRFs can be clearly seen from the figures. It can be also demonstrated by the low values of the two criteria for the goodness of the fit. Therefore, the ORFP method is well suited to extract the dynamic parameters from measured FRFs that include noise and that feature close and or distinct modes, which the RFP method using the Forsythe method has difficulty in doing as stated in [1].

6. Conclusions

The critical assessment of the Richardson's technique developed to extract modal parameters from measured FRF using the RFP model and the Forsythe method revealed several shortcomings of this technique. To overcome these shortcomings, an Optimized Rational Fraction Polynomial ORFP is proposed. It is based on the RFP model originally developed in [1] and features a constraint optimization technique to extract the modal parameters.

Several issues pertaining to programming were discussed and analyzed, including the use of different objective functions, coding and decoding of the input to the optimization process, the selection of the initial guess and the selection of the number of iterations. In addition, two criteria were introduced to assess the goodness of the fit: the mean percentage error and the maximum percentage error. Among the 6 objective functions evaluated, the mean error, the max error and the mean % error converged to an acceptable solution while the max. % error and the least square error did not converge. The mean error was found to give the best results. The initial guess of the modal parameters is suggested to be determined based on the experimental FRF, which provides the ORFP with high robustness.

The ability of the proposed technique to overcome the shortcoming of the method presented by Richardson was demonstrated through handling effectively measured FRFs that include noise, estimating of modal parameters that describe stable system. In addition, it allows the selection of the natural frequencies of the system from the experimental data, which provides the proposed technique with high flexibility and accuracy compared to the Richardson's method where the estimated natural frequencies might not correspond to the actual ones. The robustness and the effectiveness of the ORFP is demonstrated through different case studies where the measured FRFs feature close natural modes, distinct natural modes and both close and distinct natural modes over small (500 up to 2000Hz) and wide (500 up to 6000Hz) frequency spectrums.

As future work, this technique needs to be tried on receptances obtained directly from displacement measurement devices such as potentiometers or proximity sensors and compare the fitted modal parameters to those derived from receptances obtained indirectly from the integration of the

accelerance.

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O. Omar - B.Sc. Cairo University, Giza, Egypt, 1998. M.Sc. McMaster University, Hamilton, ON, Canada, 2003. PhD candidate in the Department of Mechanical Engineering, McMaster University. Research areas are in high speed machining, metal cutting, system dynamics, modal analysis, surface generation, end-milling, and optimization. Currently works as a Mechanical Designer, EIT, JNE Consulting Ltd., Hamilton, ON.